Bipolare Kurven

Cassini’sche Kurven $r' * r = k^2$

In[1]:= $f[r\_] := \frac{k^2}{r}$

In[4]:= `cas = Eliminate[{(x + e)^2 + y^2 == f[r]^2, (x - e)^2 + y^2 == r^2, r' == f[r]}, {r, r'}] // Simplify`
eliminiere … vereinfache

Out[4]= $e^4 + (x^2 + y^2)^2 == k^4 + 2 e^2 (x^2 - y^2)$

Direkt die y-Achsen Schnitthöhe ausrechnen

In[11]:= `cas /. x → 0`

Out[11]= $e^4 + y^4 == k^4 - 2 e^2 y^2$

In[17]:= `Solve[e^4 + y^4 == k^4 - 2 e^2 y^2, {y}]`
löse

Out[17]= $\{\{y \to -\sqrt{-e^2 - k^2}\}, \{y \to \sqrt{-e^2 - k^2}\}, \{y \to -\sqrt{-e^2 + k^2}\}, \{y \to \sqrt{-e^2 + k^2}\}\}$

Die y-Achse wird in der Höhe $h^2 = k^2 - e^2$ geschnitten. Existiert nur für k>=e.
k=e folgt h=0, Lemniskate

In diese Höhe lege ich eine waagerechte Gerade

In[18]:= `cas /. y → ` $\sqrt{-e^2 + k^2}$

Out[18]= $e^4 + (-e^2 + k^2 + x^2)^2 == k^4 + 2 e^2 (e^2 - k^2 + x^2)$

In[19]:= `Solve[e^4 + (-e^2 + k^2 + x^2)^2 == k^4 + 2 e^2 (e^2 - k^2 + x^2), {x}]`
löse

Out[19]= $\{\{x \to 0\}, \{x \to 0\}, \{x \to -\sqrt{2}\sqrt{2 e^2 - k^2}\}, \{x \to \sqrt{2}\sqrt{2 e^2 - k^2}\}\}$

diese schneidet für $2 e^2 > k^2$ zwei weitere Male, das sind dann die eingeschnürten Kurven
für $2 e^2 = k^2$ entfällt die Einschnürung, die flache Cassini’sche Kurve hat dann die Gleichung

In[26]:= $2 e^2 - e^2$

Out[26]= e^2

Der Flachpunkt ist (0,+- e)

die flache Cassini’sche Kurve hat dann die Gleichung

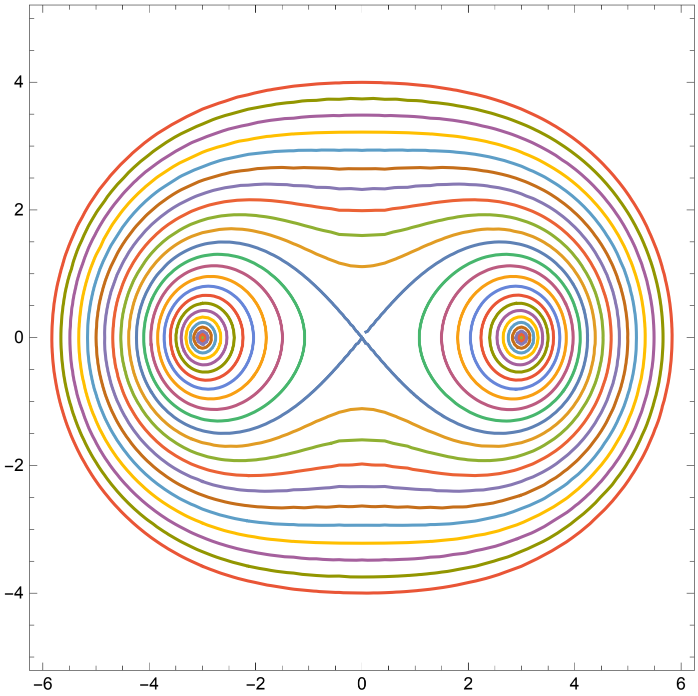
In[22]:= `cas /. k → ` $\sqrt{2}$ `e // Simplify`
vereinfache

In[18]:=

```
ContourPlot[Table[3^4 + (x^2 + y^2)^2 == k^4 + 2 × 3^2 (x^2 - y^2), {k, 0, 5, 0.2}] // Evaluate,
 {x, -6, 6}, {y, -5, 5}]
```

Konturgraphik   Tabelle   werte aus

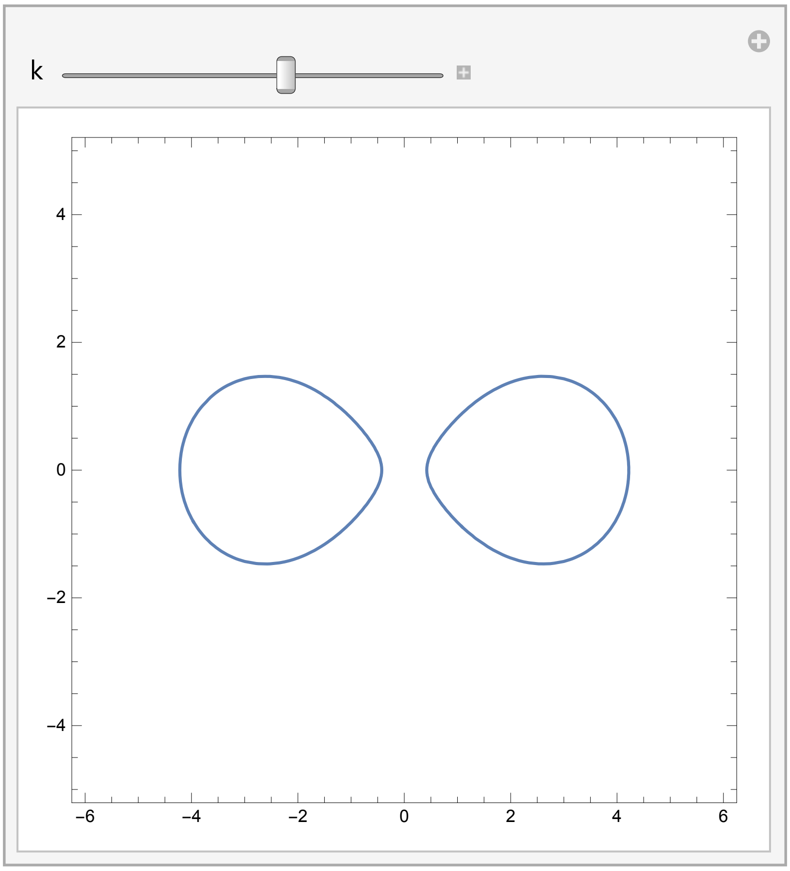
Out[18]=



```
Manipulate[
ContourPlot[{3^4 + (x^2 + y^2)^2 == k^4 + 2 × 3^2 (x^2 - y^2)}, {x, -6, 6}, {y, -5, 5}], {{k, 5}, 0, 5}]
```

manipuliere
Konturgraphik

Out[2]=



Hyperbeln r’=r+k

```
f[r_] := r + k
hyp = Eliminate[{(x + e)^2 + y^2 == f[r]^2, (x - e)^2 + y^2 == r^2, r' == f[r]}, {r, r'}]
```

eliminiere

$k^4 + k^2 \left(-4 e^2 - 4 x^2 - 4 y^2\right) == -16 e^2 x^2$

```
hyp // FullSimplify
```

vereinfache vollständig

$\left(-4 e^2 + k^2\right) \left(k^2 - 4 x^2\right) == 4 k^2 y^2$

Ellipsen r’=-r+k

```
f[r_] := -r + k
elli = Eliminate[{(x + e)^2 + y^2 == f[r]^2, (x - e)^2 + y^2 == r^2, r' == f[r]}, {r, r'}]
```

eliminiere

$k^4 + k^2 \left(-4 e^2 - 4 x^2 - 4 y^2\right) == -16 e^2 x^2$

```
elli // FullSimplify
```
vereinfache vollständig

$$\left(-4\,e^2+k^2\right)\left(k^2-4\,x^2\right)==4\,k^2\,y^2$$

Descartes'sche Ovale m r + n r' == k

In[19]:=
```
des = m r + n r' == k
```

Out[19]= $m\,r+n\,r' == k$

In[20]:=
```
desc = Eliminate[
   {(x + e)^2 + y^2 == (r')^2, (x - e)^2 + y^2 == r^2, m r + n r' == k}, {r, r'}] // FullSimplify
```
eliminiere / vereinfache vollständig

Out[20]= $$k^4+\left(e^2\,(m-n)\,(m+n)-2\,e\left(m^2+n^2\right)x+(m-n)\,(m+n)\left(x^2+y^2\right)\right)^2==2\,k^2\left(e^2\left(m^2+n^2\right)+2\,e\left(-m^2+n^2\right)x+\left(m^2+n^2\right)\left(x^2+y^2\right)\right)$$

Nullstellen

```
desc
```

$$k^4+\left(e^2\,(m-n)\,(m+n)-2\,e\left(m^2+n^2\right)x+(m-n)\,(m+n)\left(x^2+y^2\right)\right)^2==2\,k^2\left(e^2\left(m^2+n^2\right)+2\,e\left(-m^2+n^2\right)x+\left(m^2+n^2\right)\left(x^2+y^2\right)\right)$$

```
desc /. y → 0 // Simplify
```
vereinfache

$$k^4+\left(e^2\,(m-n)\,(m+n)-2\,e\left(m^2+n^2\right)x+(m-n)\,(m+n)\,x^2\right)^2==2\,k^2\left(e^2\left(m^2+n^2\right)+2\,e\left(-m^2+n^2\right)x+\left(m^2+n^2\right)x^2\right)$$

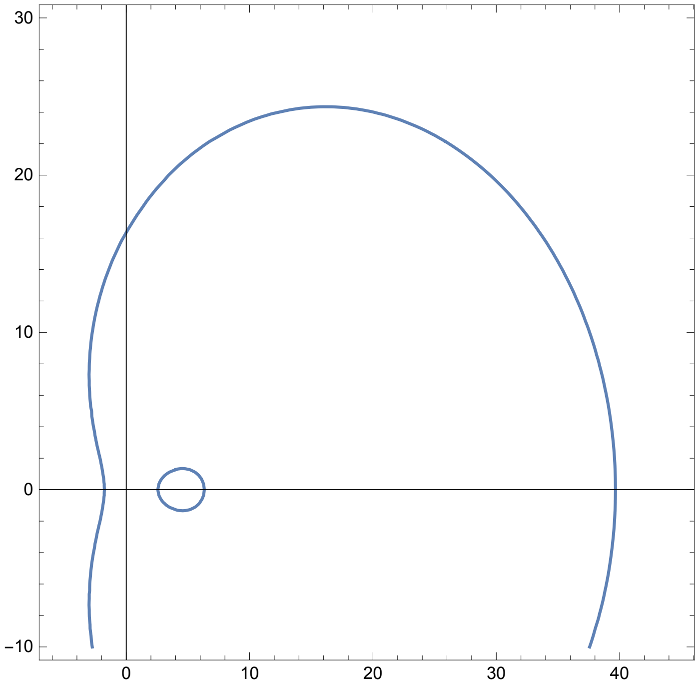
```
Solve[k^4 + (e^2 (m - n) (m + n) - 2 e (m^2 + n^2) x + (m - n) (m + n) x^2)^2 ==
   2 k^2 (e^2 (m^2 + n^2) + 2 e (-m^2 + n^2) x + (m^2 + n^2) x^2), x] // Simplify
```
löse / vereinfache

$$\left\{\left\{x\to-\frac{k-e\,m+e\,n}{m+n}\right\},\left\{x\to\frac{k+e\,m-e\,n}{m+n}\right\},\left\{x\to\frac{-k+e\,(m+n)}{m-n}\right\},\left\{x\to\frac{k+e\,(m+n)}{m-n}\right\}\right\}$$

In[22]:=
```
we = {e = 3, k = 5, m = 1.3, n = -1}
```

Out[22]= {3, 5, 1.3, -1}

In[23]:= `ContourPlot[desc // Evaluate, {x, -6, 45}, {y, -10, 30}, Axes → True]`
Konturgraphik werte aus Axen wahr

Out[23]=


$$\left\{-\frac{k - e m + e n}{m + n}, \frac{k + e m - e n}{m + n}, \frac{-k + e (m + n)}{m - n}, \frac{k + e (m + n)}{m - n}\right\}$$

{6.33333, 39.6667, -1.78261, 2.56522}

`we = {e = 3, k = 5, m = 13 / 10, n = -1}`

$$\left\{3, 5, \frac{13}{10}, -1\right\}$$

$$xo = \frac{-k + e (m + n)}{m - n}$$

$$-\frac{41}{23}$$

`xo`
`b`

$$-\frac{41}{23}$$

1.23671

`xo - b`

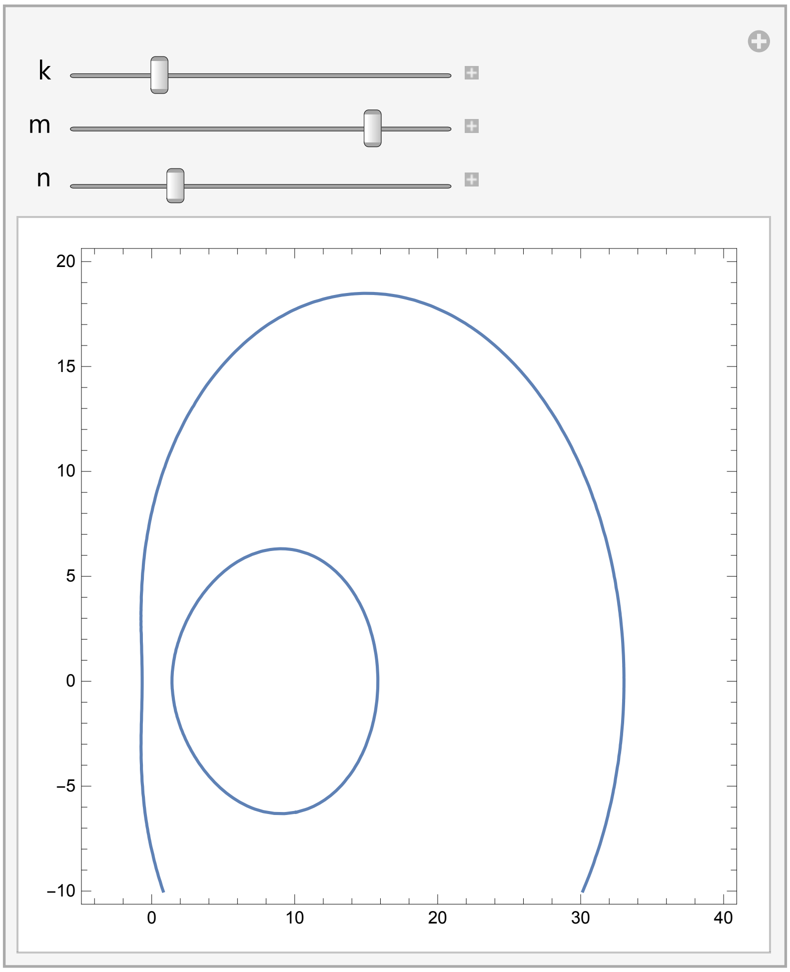
-3.01932

In[24]:= `wn = {e =., k =., m =., n =.};`

In[25]:=
```
e = 3;
Manipulate[ContourPlot[{k^4 + (e^2 (m - n) (m + n) - 2 e (m^2 + n^2) x + (m - n) (m + n) (x^2 + y^2))^2 ==
    2 k^2 (e^2 (m^2 + n^2) + 2 e (-m^2 + n^2) x + (m^2 + n^2) (x^2 + y^2)), x == xo - b},
  {x, -4, 40}, {y, -10, 20}], {{k, 5}, 1, 8}, {{m, 1.3}, -2, 2}, {{n, -1}, -2, 2}]
```

manipuliere Konturgraphik

Out[26]=



Suche nach der Beule

```
e =.; xo = (-k + e (m + n))/(m - n)
```

$$\frac{-k + e\ (m + n)}{m - n}$$

senkrechte Gerade, links von xo ist x==xo-b

$$\{3,\ 5,\ \frac{13}{10},\ -1\}$$

$$\{3,\ 5,\ \frac{13}{10},\ -1\}$$

```
desc
```

$$k^4 + \left(e^2 (m-n)(m+n) - 2 e \left(m^2+n^2\right) x + (m-n)(m+n)\left(x^2+y^2\right)\right)^2 == 2 k^2 \left(e^2 \left(m^2+n^2\right) + 2 e \left(-m^2+n^2\right) x + \left(m^2+n^2\right)\left(x^2+y^2\right)\right)$$

```
we = {e = 3, k = 5, m = 1.3, n = -1}
{3, 5, 1.3, -1}

b =.;

test = desc /. {x → xo - b}
```

$$625 + \left(6.21 - 16.14 (-1.78261 - b) + 0.69 \left((-1.78261 - b)^2 + y^2\right)\right)^2 == 50 \left(24.21 - 4.14 (-1.78261 - b) + 2.69 \left((-1.78261 - b)^2 + y^2\right)\right)$$

```
Solve[test, y] // Simplify // N
löse                vereinfache     numerischer Wert
```

$$\{\{y \to -0.0144928 \sqrt{416000. - 26000. \sqrt{256. - 207. b} - 128340. b - 4761. b^2}\},$$
$$\{y \to 0.0144928 \sqrt{416000. - 26000. \sqrt{256. - 207. b} - 128340. b - 4761. b^2}\},$$
$$\{y \to -0.0144928 \sqrt{416000. + 26000. \sqrt{256. - 207. b} - 128340. b - 4761. b^2}\},$$
$$\{y \to 0.0144928 \sqrt{416000. + 26000. \sqrt{256. - 207. b} - 128340. b - 4761. b^2}\}\}$$

```
diskrimi = 256 - 207 b == 0;

Solve[diskrimi, b] // N
löse                numerischer Wert

{{b = 1.2367149758454106`}}

{{1.23671}}

b
1.23671
```

b passt zur Zeichnung, Ordinate dazu

```
0.014492753623188406` √(416000.` + 26000.` √(256.` - 207.` b) - 128340.` b - 4761.` b^2)
7.24635

desc /. {x → xo - b} (* nochmal direkt *)
```

$$625 + \left(54.9419 + 0.69 \left(9.11632 + y^2\right)\right)^2 == 50 \left(36.71 + 2.69 \left(9.11632 + y^2\right)\right)$$

```
Solve[625 + (54.9419 + 0.69 (9.11632 + y^2))^2 == 50 (36.71 + 2.69 (9.11632 + y^2)), {y}]
löse
{{y → -7.24635}, {y → -7.24635}, {y → 7.24635}, {y → 7.24635}}
```

Das passt. Die Beule ist nur im konkreten Fall bestimmt.

```
we = {e = 3, k = 5, m = 1.3, n = -1}
{3, 5, 1.3, -1}

wn = {e =., k =., m =., n =.};
```

Experimente

```
desc
```

$$k^4 + \left(e^2 (m-n)(m+n) - 2 e \left(m^2+n^2\right) x + (m-n)(m+n)\left(x^2+y^2\right)\right)^2 = 2 k^2 \left(e^2 \left(m^2+n^2\right) + 2 e \left(-m^2+n^2\right) x + \left(m^2+n^2\right)\left(x^2+y^2\right)\right)$$

```
descl = k^4 + (e^2 (m^2 - n^2) - 2 e (m^2 + n^2) x + (m^2 - n^2) (x^2 + y^2))^2 -
   2 k^2 (e^2 (m^2 + n^2) - 2 e (m^2 - n^2) x + (m^2 + n^2) (x^2 + y^2))
```

$$k^4 + \left(e^2 \left(m^2-n^2\right) - 2 e \left(m^2+n^2\right) x + \left(m^2-n^2\right)\left(x^2+y^2\right)\right)^2 - 2 k^2 \left(e^2 \left(m^2+n^2\right) - 2 e \left(m^2-n^2\right) x + \left(m^2+n^2\right)\left(x^2+y^2\right)\right)$$

```
we = {e = 10, k = 4, m = 1/2, n = 2}
```

$$\left\{10, 4, \frac{1}{2}, 2\right\}$$

```
descl =
  256 + (375 + 85 x + 15/4 (x^2 + y^2))^2 - 32 (425 + 75 x + 17/4 (x^2 + y^2)) // FullSimplify // Expand
```

vereinfache vollstä··· multiplizie

$$127281 + 61350\, x + \frac{19803\, x^2}{2} + \frac{1275\, x^3}{2} + \frac{225\, x^4}{16} + \frac{5353\, y^2}{2} + \frac{1275\, x\, y^2}{2} + \frac{225\, x^2 y^2}{8} + \frac{225\, y^4}{16}$$

```
256 + (375 + 85 x + 15/4 (x^2 + y^2))^2 - 32 (425 + 75 x + 17/4 (x^2 + y^2))
```

$$256 + \left(375 + 85\, x + \frac{15}{4}\left(x^2+y^2\right)\right)^2 - 32\left(425 + 75\, x + \frac{17}{4}\left(x^2+y^2\right)\right)$$

Dieses sind jedenfalls keine Kreise!

```
wn = {e =., k =., m =., n =.};
```